J. DIFFERENTIAL GEOMETRY
74 (2006) 265-292

FRACTIONAL ANALYTIC INDEX

V. Mathai, R.B. Melrose & I.M. Singer

Abstract

For a finite rank projective bundle over a compact manifold, so associated to a torsion, Dixmier-Douady, 3-class, w, on the manifold, we define the ring of differential operators 'acting on sections of the projective bundle' in a formal sense. In particular, any oriented even-dimensional manifold carries a projective spin Dirac operator in this sense. More generally the corresponding space of pseudodifferential operators is defined, with supports sufficiently close to the diagonal, i.e., the identity relation. For such elliptic operators we define the numerical index in an essentially analytic way, as the trace of the commutator of the operator and a parametrix and show that this is homotopy invariant. Using the heat kernel method for the twisted, projective spin Dirac operator, we show that this index is given by the usual formula, now in terms of the twisted Chern character of the symbol, which in this case defines an element of K-theory twisted by w; hence the index is a rational number but in general it is not an integer.

The Atiyah-Singer index theorem for an elliptic (pseudodifferential) operator gives an integrality theorem; namely a certain characteristic integral is an integer because it is the index of an elliptic operator. Notably, for a closed spin manifold Z, the $\widehat{A}$ genus, $\int_Z \widehat{A}(Z)$ is an integer because it is equal to the index of the Dirac operator on Z. When Z is not a spin manifold, the spin bundle S does not exist, as a vector bundle, and when Z has no $\text{spin}_{\mathbb{C}}$ structure, there is no global vector bundle resulting from the patching of the local bundles $S \otimes \mathcal{L}_i$, where the $\mathcal{L}_i$ are line bundles. However, as we show below, S is a always a projective vector bundle associated to the the Clifford algebra $\mathbb{C}\mathrm{l}(Z)$, of the cotangent bundle T^*Z which is an Azumaya bundle cf. [**10**]. Such a (finite rank) projective vector bundle, E, over a compact manifold has local trivializations which may fail to satisfy the cocycle condition on triple overlaps by a scalar factor; this defines the Dixmier-Douady invariant in $H^3(Z,\mathbb{Z})$. If this torsion twisting is non-trivial there is no, locally spanning, space of global sections. The Dixmier-Douady invariant for $\mathbb{C}\mathrm{l}(Z)$ is the third integral Stieffel-Whitney class, $W_3(Z)$. In particular, the spin Dirac operator does not exist when Z is not a spin manifold.

Received 01/10/2006.

Correspondingly the $\widehat{A}$ genus is a rational number, but not necessarily an integer. In this paper we show that, in the oriented even-dimensional case, one can nevertheless define a projective spin Dirac operator, with an analytic index valued in the rational numbers, and prove the analogue of the Atiyah-Singer index theorem for this operator twisted by a general projective bundle. In fact we establish the analogue of the Atiyah-Singer index theorem for a general projective elliptic pseudodifferential operator. In a subsequent paper the families case will be discussed.

For a compact manifold, Z, and vector bundles E and F over Z the Schwartz kernel theorem gives a one-to-one correspondence between continuous linear operators from $\mathcal{C}^\infty(Z,E)$ to $\mathcal{C}^{-\infty}(Z,F)$ and distributions in $\mathcal{C}^{-\infty}(Z^2,\operatorname{Hom}(E,F)\otimes\Omega_R)$. Here $\operatorname{Hom}(E,F)$ is the 'big' homomorphism bundle over Z^2 with fibre at (z,z') equal to $\hom(E_{z'},F_z)\equiv F_z\otimes E'_{z'}$, and Ω_R is the density bundle lifted from the right factor. Restricted to pseudodifferential operators of order m, this becomes an isomorphism to the space, $I^m(Z^2,\operatorname{Diag};\operatorname{Hom}(E,F)\otimes\Omega_R)$, of conormal distributions with respect to the diagonal, cf. [**7**].

This fact motivates our definition of projective pseudodifferential operators when E and F are only projective vector bundles associated to a fixed finite-dimensional Azumaya bundle $\mathcal{A}$. The homomorphism bundle $\operatorname{Hom}(E,F)$ is then again a projective bundle on Z^2 associated to the tensor product $\mathcal{A}_L\otimes\mathcal{A}'_R$ of the pull-back of $\mathcal{A}$ from the left and the conjugate bundle from the right. In particular if E and F have DD invariant $\tau\in H^3(Z;\mathbb{Z})$ then $\operatorname{Hom}(E,F)$ has DD invariant $\pi_L^*\tau-\pi_R^*\tau\in H^3(Z^2;\mathbb{Z})$. Since this class is trivial in a tubular neighborhood of the diagonal it is reasonable to expect that $\operatorname{Hom}(E,F)$ may be realized as an ordinary vector bundle there. In fact this is the case and there is a canonical choice, $\operatorname{Hom}^{\mathcal{A}}(E,F)$ of extension. This allows us to identify the space of projective pseudodifferential operators, with kernels supported in a sufficiently small neighborhood N_ϵ of the diagonal, $\Psi^\bullet_\epsilon(Z;E,F)$ with the space of conormal distributions $I^\bullet_\epsilon(N_\epsilon,\operatorname{Diag};\operatorname{Hom}^{\mathcal{A}}(E,F)\otimes\Omega_R)$. Despite not being a space of operators, this has precisely the same local structure as in the untwisted case and has similar composition properties provided supports are restricted to appropriate neighborhoods of the diagonal. The space of projective smoothing operators $\Psi^{-\infty}_\epsilon(Z;E,F)$ is therefore identified with $\mathcal{C}^\infty_{\mathrm{c}}(N_\epsilon;\operatorname{Hom}^{\mathcal{A}}(E,F)\otimes\pi_R^*\Omega)$. The principal symbol map is well defined for conormal distributions so this leads directly to the symbol map on $\Psi^m_\epsilon(Z;E,F)$ with values in smooth homogeneous sections of degree m of $\hom(E,F)$, the 'little' or 'diagonal' homomorphism bundle which *is* a vector bundle. Thus ellipticity is well defined, as the invertibility of this symbol. The 'full' symbol map is given by the map to the quotient $\Psi^\bullet_\epsilon(Z;E,F)/\Psi^{-\infty}_\epsilon(Z;E,F)$. The usual calculus can then be applied and ellipticity, as invertibilty of the principal symbol, implies invertibility of the image of an operator in this quotient. Any lift, B, of the inverse is a parametrix for the given elliptic operator, A. The analytic index of the projective elliptic operator is then defined by

$$\operatorname{ind}_{\mathrm{a}}(A) = \operatorname{Tr}(AB - \operatorname{Id}_F) - \operatorname{Tr}(BA - \operatorname{Id}_E) = \operatorname{Tr}([A, B]) \tag{1}$$

where the last expression, though compact, is slightly misleading. Directly from this formula it appears that $\operatorname{ind}_{\mathrm{a}}(A)$ might be complex valued. Using the homotopy invariance discussed below it can be see directly to be real; from the index formula it follows that $\operatorname{ind}_{\mathrm{a}}(A)$ is rational.

In fact we further extend this discussion by allowing twisting by line bundles defined over the bundle of trivializations of the Azumaya bundle; these are Nth roots of line bundles over the base and have actions of $\mathrm{SU}(N)$ arising from fibre-holonomy. This allows us to include in the same general framework the case of $\mathrm{spin}_{\mathbb{C}}$ Dirac operators on non-spin manifolds. Thus we define projective $\mathrm{spin}_{\mathbb{C}}$ Dirac operators even when there is no $\mathrm{spin}_{\mathbb{C}}$ structure.

Within the projective pseudodifferential operators, acting between two projective bundles associated to the same Azumaya bundle, there is always a full algebra of differential operators, with kernels supported within the diagonal. On an even-dimensional oriented manifold the Clifford bundle is an Azumaya bundle and has associated to it projective spin bundles, $S^{\pm}$. The choice of a compatible connection gives a projective spin Dirac operator. Such a projective Dirac operator, $\eth$, can be coupled to any unitary projective vector bundle E over Z associated to an Azumaya algebra $\mathcal{A}$. Thus $S \otimes E$ is a projective vector bundle associated to the Azumaya algebra $\mathbb{C}\mathrm{l}(Z) \otimes \mathcal{A}$. This coupled operator, $\eth^+_E$, is elliptic and its analytic index, in the sense defined above, is

$$\int_Z \widehat{A}(Z) \wedge \operatorname{Ch}_{\mathcal{A}}(E)$$

where $\operatorname{Ch}_{\mathcal{A}} : K^0(Z, \mathcal{A}) \to H^{even}(Z, \mathbb{Q})$ is the twisted Chern character.

When Z is even dimensional, $K^0_{\mathrm{c}}(T^*Z, \pi^*\mathcal{A}) \otimes \mathbb{Q}$ is generated by the classes of symbols of such coupled signature operators. We conclude from this, essentially as in the untwisted case, that for a general projective elliptic pseudodifferential operators $T \in \Psi^{\bullet}_{\epsilon}(Z; E, F)$ with principal symbol $\sigma_m(T)$,

$$\operatorname{ind}_{\mathrm{a}}(T) = \int_{T^*Z} \operatorname{Td}(T^*Z) \wedge \operatorname{Ch}_{\mathcal{A}}(\sigma_m(T)).$$

This in turn shows the rationality of the analytic index and we conclude by providing several examples where ind_a is not an integer, but only a fraction, as justification for the title of the paper.

In the first section below the 'big' homomorphism bundle is constructed, near the diagonal, for any two projective bundles associated to the same Azumaya bundle. In Section 2 we discuss a more general construction of such 'big' homomorphism bundles which corresponds to twisting by a line bundle, only the Nth power of which is well-defined over the base. The projective spin bundle is discussed in the next section as is its relationship to the spin bundle when a spin structure exists and to $\mathrm{spin}_{\mathbb{C}}$ bundles when they exist. In Sections 4 and 5 the notion of smoothing, and then general pseudodifferential, operators between projective bundles (and more generally twisted projective bundles) is introduced and for elliptic operators the index is defined. The homotopy invariance of the index is shown directly in Section 6 and in Section 7 projective Dirac operators are defined and the usual local index formula is used to compute the index in that case. Much as in the usual case this formula is extended to general pseudodifferential operators in Section 8. Some examples in which the index is truly fractional are given in the final section.

1. Homomorphism bundles

Let $\mathcal{A}$ be a finite-dimensional (star) Azumaya bundle over a compact manifold Z; see [**10**] for more details. By definition $\mathcal{A}$ is a complex vector bundle over Z with fibres having algebra structures and with local (algebra) trivializations as $N \times N$ matrix algebras. Since the automorphism group of the star-algebra of $N \times N$ matrices is $\mathrm{PU}(N)$ (acting by conjugation) the bundle of all such trivializations, $\mathcal{P}$, is a principal $\mathrm{PU}(N)$-bundle. From the Azumaya perspective, the 'trivial' case is where $\mathcal{A} \otimes \hom(A_1) = \hom(A_2)$ is 'stably' the homomorphism bundle of a vector bundle over Z and this corresponds to the existence of a stable lifting of $\mathcal{P}$ to a $\mathrm{U}(M)$-principal bundle.

A projective vector bundle, E, over Z can be defined (a different initial approach is taken in [**10**]) as a projection-valued section of $\mathcal{A} \otimes \mathcal{K}$, for the algebra $\mathcal{K}$ of compact operators on some Hilbert space $\mathcal{H}$. Any projection in $\mathcal{A}_z \otimes \mathcal{K}$ is of finite rank, so over a set in which $\mathcal{A}$ is trivial this yields a vector bundle. However, the phases of the transition maps between trivializations are not determined, and cannot, in general, be chosen to satisfy the cocycle condition, so in general these are not vector bundles. The transpose Azumaya bundle $\mathcal{A}^t$ is $\mathcal{A}$ with multiplication reversed and $\mathcal{A} \otimes \mathcal{A}^t$ is trivial as an Azumaya bundle, since it has structure group acting through the adjoint representation, $\mathrm{PU}(N) \longrightarrow \mathrm{PU}(N^2)$, which lifts canonically to a $\mathrm{U}(N)$ action. For any two projective vector bundles E and F associated to $\mathcal{A}$ it follows that $\hom(E, F)$, since it is associated to $\mathcal{A} \otimes \mathcal{A}^t$, is a true vector bundle.

The lift of $\mathcal{A}$ to an Azumaya bundle over $\mathcal{P}$ is trivial, i.e., is a homomorphism bundle, and correspondingly the lift of a projective vector bundle E associated to $\mathcal{A}$ to $\mathcal{P}$ is a finite-dimensional subbundle, $\tilde{E} \subset \mathbb{C}^N \otimes \mathcal{H}$, over $\mathcal{P}$ which is equivariant for the standard action of

$\mathrm{U}(N)$ on $\mathbb{C}^N$, interpreted as covering the $\mathrm{PU}(N)$ action on $\mathcal{P}$. Since the action of $\mathrm{U}(N)$ on $\hom(\tilde{E}, \tilde{F})$, for the lifts of any two projective vector bundles associated to $\mathcal{A}$, is through conjugation we see that this is a bundle over $\mathcal{P}$ invariant under the $\mathrm{PU}(N)$ action and hence, again, we see that it descends to $\hom(E, F)$, as a well-defined vector bundle on Z. On the other hand the 'big' homomorphism bundle $\mathrm{Hom}(\tilde{E}, \tilde{F})$ is only a projective vector bundle over Z^2; it is associated to the *external* tensor product $\mathcal{A} \boxtimes \mathcal{A}^t$ over Z^2. Since at the diagonal it is a vector bundle, reducing there to $\hom(E, F)$ it is reasonable to expect it be represented by a vector bundle in a neighbourhood of the diagonal. For our purposes it is vitally important that this extension be made in such a way that the composition properties also extend.

For a given metric on Z set

$$N_\epsilon = \{(z, z') \in Z^2; d_g(z, z') < \epsilon\}. \tag{2}$$

The projective unitary group, $\mathrm{PU}(N)$, can be written as a quotient of the group, $\mathrm{SU}(N)$, of unitary matrices of determinant one:

$$\mathbb{Z}_N \longrightarrow \mathrm{SU}(N) \longrightarrow \mathrm{PU}(N). \tag{3}$$

In the following result, which is the foundation of subsequent developments, we use the discreteness of the fibres of (3).

Proposition 1. *Given two projective bundles, E and F, associated to a fixed Azumaya bundle and $\epsilon > 0$ sufficiently small, the exterior homomorphism bundle* $\mathrm{Hom}(\tilde{E}, \tilde{F})$, *descends from a neighborhood of the diagonal in $\mathcal{P} \times \mathcal{P}$ to a vector bundle,* $\mathrm{Hom}^{\mathcal{A}}(E, F)$, *over N_ϵ extending* $\hom(E, F)$. *For any three such bundles there is a natural associative composition law*

$$\begin{aligned}\mathrm{Hom}^{\mathcal{A}}_{(z'', z')}(F, G) \times \mathrm{Hom}^{\mathcal{A}}_{(z, z'')}(E, F) \ni (a, a') \\ \longmapsto a \circ a' \in \mathrm{Hom}^{\mathcal{A}}_{(z, z')}(E, G),\ (z, z''), (z'', z') \in N_{\epsilon/2}\end{aligned} \tag{4}$$

which is consistent with the composition over the diagonal.

Remark 1. Applying this result to the projective vector bundle given by $\mathrm{Id}_{\mathcal{A}} \otimes \pi_1$, where π_1 is the projection onto the first basis element of $\mathcal{H}$, gives a bundle, which we denote $\widehat{\mathcal{A}}$, which extends $\mathcal{A}$ from the diagonal to some neighborhood N_ϵ and which has the composition property as in (4)

$$\widehat{\mathcal{A}}_{(z'', z)} \times \widehat{\mathcal{A}}_{(z, z'')} \longrightarrow \widehat{\mathcal{A}}_{(z, z')}. \tag{5}$$

We regard this as the natural extension of $\mathcal{A}$.

Proof. Consider again the construction of $\hom(E, F)$, always for two projective bundles associated to the same Azumaya bundle, $\mathcal{A}$. The dual bundle $\tilde{E}'$ is associated to the adjoint Azumaya bundle; or as a subbundle of $\mathbb{C}^N \otimes \mathcal{H}$ over $\mathcal{P}$ it is associated with the adjoint action of

$\mathrm{SU}(N)$ on $\mathbb{C}^N$. The external tensor product over the product of $\mathcal{P}$ with itself, as a bundle over Z^2, $\tilde{F}\boxtimes\tilde{E}'$ is therefore a subbundle of $\hom(\mathbb{C}^N)\otimes\hom(\mathcal{H})$ over $\mathcal{P}\times\mathcal{P}$ equivariant for the action of $\mathrm{SU}(N)\times\mathrm{SU}(N)$ over $\mathrm{PU}(N)\times\mathrm{PU}(N)$. Restricted to the diagonal, $\mathcal{P}\times\mathcal{P}$ has the natural diagonal subbundle $\mathcal{P}$. The restriction of $\tilde{F}\boxtimes\tilde{E}'$ to this submanifold has a $\mathrm{PU}(N)$ action, and so descends to the bundle $\hom(E,F)$ over $\mathrm{Diag}\equiv Z$, since for $A\in\mathrm{PU}(N)$ we can take the same lift $\tilde{A}$ to $\mathrm{SU}(N)$ in each factor and these different diagonal lifts lead to the same operator through conjugation.

In a neighborhood, N_ϵ, of the diagonal there is a corresponding 'near diagonal' submanifold of $\mathcal{P}\times\mathcal{P}$; for instance we can extend $\mathcal{P}$ over the diagonal to a subbundle $\widetilde{\mathcal{P}}\subset\mathcal{P}\times\mathcal{P}$ by parallel transport normal to the diagonal for some connection on $\mathcal{A}$. Now, any two points of $\widetilde{\mathcal{P}}$ in the same fibre over a point in N_ϵ are related by the action of $(A',A)\in\mathrm{PU}(N)\times\mathrm{PU}(N)$ where $A'A^{-1}$ is in a fixed small neighborhood of the diagonal, only depending on ϵ. It follows from the discreteness of the quotient $\mathrm{SU}(N)\longrightarrow\mathrm{PU}(N)$ that, for $\epsilon>0$ sufficiently small, on lifting A to $\tilde{A}\in\mathrm{SU}(N)$ there is a unique neighboring lift, $\tilde{A}'$, of A'. The conjugation action of these lifts on $\tilde{F}\boxtimes\tilde{E}'$ is therefore independent of choices, so defining $\mathrm{Hom}^{\mathcal{A}}(E,F)$ over N_ϵ. This bundle certainly restricts to $\hom(E,F)$ over the diagonal.

In fact this construction is independent of the precise choice of $\widetilde{\mathcal{P}}$. Namely if $\mathrm{Hom}(\tilde{E},\tilde{F})\equiv\tilde{F}\boxtimes\tilde{E}'$ is restricted to a sufficiently small open neighborhood, N, of $\mathcal{P}$ as the diagonal in $\mathcal{P}\times\mathcal{P}$, then the part of $\mathrm{PU}(N)\times\mathrm{PU}(N)$ acting on the fibres of N lifts to act linearly on $\mathrm{Hom}(\tilde{E},\tilde{F})$, so defining $\mathrm{Hom}^{\mathcal{A}}(E,F)$ as a bundle over the projection of N into Z^2. It follows that this action is consistent with the composition of $\mathrm{Hom}(\tilde{E},\tilde{F})$ and $\mathrm{Hom}(\tilde{F},\tilde{G})$ for any three projective bundles associated to $\mathcal{A}$. This leads to the composition property (4). q.e.d.

As a bundle over $\mathcal{P}$, the projective bundle $\tilde{E}$ can be given an $\mathrm{SU}(N)$ invariant connection. A choice of such connections on $\tilde{E}$ and $\tilde{F}$ induces, as in the standard case, a connection on $\mathrm{Hom}(\tilde{E},\tilde{F})$ over $\mathcal{P}\times\mathcal{P}$ and hence a connection on $\mathrm{Hom}^{\mathcal{A}}(E,F)$ over N_ϵ.

Remark 2. Since $\mathcal{H}$ is a fixed Hilbert space we can also identify $\mathcal{K}$ as a trivial bundle over Z^2. The construction about then identifies $\mathrm{Hom}^{\mathcal{A}}(E,F)$ as a subbundle of $\widehat{\mathcal{A}}\otimes\mathcal{K}$, as a bundle over N_ϵ, with the composition (4) induced from (5).

Remark 3. A particularly important case of an Azumaya bundle is the Clifford bundle on any oriented even-dimensional manifold (in the odd-dimensional case the complexified Clifford bundle is not quite an Azumaya bundle but rather the direct sum of two). On a manifold of dimension $2n$ this is locally isomorphic to the algebra of $2^n\times 2^n$

matrices. Letting $\mathcal{P}_{\mathbb{C}\mathrm{l}}$ be the associated principal $\mathrm{PU}(2^n)$-bundle of trivializations, we call the trivial bundle, $\mathbb{C}^{2^n}$, over $\mathcal{P}_{\mathbb{C}\mathrm{l}}$ the *projective spin bundle*; the relationship to the usual spin bundle is explained in Section 3. Proposition 1 and Remark 1, give an extension of the Clifford bundle to a bundle, $\widehat{\mathbb{C}\mathrm{l}}$, in neighborhood of the diagonal as the 'big' homomorphism bundle of this projective spin bundle. The discussion below shows that this allows us to define a projective spin Dirac operator even when no spin (or even $\mathrm{spin}_{\mathbb{C}}$) structure exits. However, it is an element of an algebra of 'differential operators' which does not have any natural action.

2. Twisting by a line bundle over $\mathcal{P}$

In Proposition 1 we have described a canonical extension of a given Azumaya algebra $\mathcal{A}$ to a bundle $\widehat{\mathcal{A}}$ near the diagonal. In general this canonical extension, the existence of which is based on the discreteness of the cover of $\mathrm{PU}(N)$ by $\mathrm{SU}(N)$, is not unique as an extension with the composition property (5). Rather, it is based on the selection, natural as it is, of the trivial bundle $\tilde{E} = \mathcal{P} \times \mathbb{C}^N$ with its natural $\mathrm{SU}(N)$ action, as generating $\mathcal{A}$ through $\hom(\tilde{E})$. In this section we consider the possibility of other choices of bundle in place of $\tilde{E}$ and hence other extensions of $\mathcal{A}$.

To motivate this discussion, consider the case of an Azumaya bundle which is trivial, in the sense that it is isomorphic to $\hom(W)$ for an Hermitian vector bundle W. The frame bundle $\mathcal{W}$ of W is a principal $\mathrm{U}(N)$ bundle to which $\hom(W)$ lifts to be the trivial bundle of $N \times N$ matrices on which $\mathrm{U}(N)$ acts through conjugation. Thus the center acts trivially, so $\hom(W)$ can also be identified with the trivial bundle of $N \times N$ matrices over $\mathcal{P} = \mathcal{W}/\,\mathrm{U}(1)$. The circle bundle, $\mathcal{L}$, over $\mathcal{P}$ with total space $\mathcal{W}$ has an induced $\mathrm{SU}(N)$ action and W can be identified with the bundle $L_W \otimes \mathbb{C}^N$ over $\mathcal{P}$, where L_W is the line bundle corresponding to $\mathcal{L}$. Abstracting this situation we arrive at the corresponding notion for a general Azumaya bundle.

Definition 1. A *representing bundle* for a star Azumaya bundle $\mathcal{A}$ is a vector bundle $\tilde{V}$ over $\mathcal{P}$ equipped with an action of $\mathrm{SU}(N)$ which is equivariant for the $\mathrm{PU}(N)$ action on $\mathcal{P}$ with the center acting as scalars and with an isomorphism, as bundles of algebras, of $\mathcal{A}$ and $\hom(\tilde{V})$ as a bundle over the base.

When appropriate we consider the unique $\mathrm{U}(N)$ action on $\tilde{V}$ for which the center also acts as scalars and such that the only elements of the center acting trivially are elements of $\mathbb{Z}_N \subset \mathrm{SU}(N)$. Note that this is consistent with the 'trivial' case discussed above.

We consider two such representing bundles, $\tilde{V}_1$, $\tilde{V}_2$, to be equivalent if there is a bundle isomorphism between them which intertwines the

$\mathrm{SU}(N)$ actions and projects to intertwine the isomorphisms with $\mathcal{A}$. To understand the non-equivalent representing bundles we study the line bundles on $\mathcal{P}$. The fibres of $\mathcal{P}$ are diffeomorphic to $\mathrm{PU}(N)$ so all line bundles over $\mathcal{P}$ have flat connections over the fibres.

Proposition 2 (cf. Kostant [**9**] and Brylinski [**3**])**.** *The total space of any line bundle on* $\mathcal{P}$ *admits a 'fibre holonomy' action by* $\mathrm{SU}(N)$ *which is equivariant for the* $\mathrm{PU}(N)$ *action on* $\mathcal{P}$*, is linear between the fibres and in which the centre acts as the fibre holonomy; this canonical* $\mathrm{SU}(N)$ *action is unique up to conjugation by a bundle isomorphism.*

As for representing bundles we consider the unique $\mathrm{U}(N)$ action on the line bundle for which the center acts also acts as scalars and such that the only elements of the center acting trivially are elements of $\mathbb{Z}_N \subset \mathrm{SU}(N)$.

Proof. Let L be a given line bundle on $\mathcal{P}$; choose some connection, ∇, on it with curvature $\omega \in \mathcal{C}^\infty(\mathcal{P};\Lambda^2)$. Each fibre of $\mathcal{P}$ is diffeomorphic to $\mathrm{PU}(N)$. Since $H^2(\mathrm{PU}(N),\mathbb{R}) = \{0\}$ the restriction of ω to each fibre is exact. Thus we can find a smooth 1-form, α, on $\mathcal{P}$ such that $\omega - d\alpha$ vanishes on each fibre. It follows that the connection $\nabla - \alpha$ has vanishing curvature on each fibre of $\mathcal{P}$.

Now, the $\mathrm{SU}(N)$ action on L is given by parallel transport with respect to such a connection. For each smooth curve $c : [0,1] \longrightarrow \mathrm{PU}(N)$ with $c(0) = \mathrm{Id}$ and each point $l \in L_p$ consider the curve $c(t)p$ in the fibre through $p \in \mathcal{P}$ and let $s(l) \in L_{c(1)p}$ be obtained by parallel transport along L over $c(t)p$. This certainly gives a smooth map $s(c)$ on the total space of L which is linear on the fibres. Furthermore, since the curvature on each fibre vanishes, $s(c)$ depends only on the homotopy class of c in $\mathrm{PU}(N)$ as a curve from Id to $g = c(1) \in \mathrm{PU}(N)$ and composition of curves leads to the composite map. Thus in fact s is an action of the universal covering group, $\mathrm{SU}(N)$, of $\mathrm{PU}(N)$ on the total space of L as desired. The centre, $\mathbb{Z}_N$, of $\mathrm{SU}(N)$ gives the fibre holonomy essentially by definition.

Any two connections on L which are fibre-flat differ by a 1-form β which is closed on each fibre. Again, since $H^1(\mathrm{PU}(N),\mathbb{R}) = \{0\}$, we may choose $f \in \mathcal{C}^\infty(\mathcal{P})$ such that $\beta - df$ vanishes on each fibre. Parallel transport along curves in $\mathrm{PU}(N)$ as discussed above, for the two connections, is then intertwined by the bundle isomorphism $\exp(f)$. Thus the $\mathrm{SU}(N)$ action defined by parallel transport on the fibres is well-defined up to bundle isomorphism. q.e.d.

Remark 4. An alternate proof of Proposition 2 uses Cheeger-Simons characters, and will be described here. As in the proof of Proposition 2,

1) given a line bundle L over $\mathcal{P}$, we can always find a connection with curvature F with the property that F restricted to the fibers is trivial.
2) $\mathrm{SU}(N)$-actions on L that cover the $\mathrm{PU}(N)$ action on $\mathcal{P}$ are obtained from characters of $\mathbb{Z}_N$ via the holonomy of flat connections on line bundles along $\mathrm{PU}(N)$.
3) Consider the exact sequence

$$0 \to H^1(\mathcal{P}, \mathbb{R})/H^1(\mathcal{P}, \mathbb{Z}) \to \check{H}^2(\mathcal{P}) \to A^2(\mathcal{P}) \to 0 \tag{6}$$

where $\check{H}^2(\mathcal{P})$ denotes the Cheeger-Simons characters of 1-cycles on $\mathcal{P}$ and

$$A^2(\mathcal{P}) = \Big\{(c_1, F); \text{where } F \text{ is a closed 2-form on } \mathcal{P} \text{ representing } c_1, \\ \text{which is in the image of } H^2(\mathcal{P}, \mathbb{Z}) \text{ in } H^2(\mathcal{P}, \mathbb{R})\Big\},$$

cf. p. 25, middle formula of equation (3.3), in [**6**].

Take any pair $(c_1(L), F)$ as in (1) above. Then there exists a Cheeger-Simons character $\chi : Z_1(\mathcal{P}) \to \mathbb{R}/\mathbb{Z}$, whose value on a closed curve is the holonomy of some connection with curvature equal to F. Now the exact sequence (6) when restricted to any fiber of $\mathcal{P}$ reduces to,

$$0 \to \check{H}^2(\mathrm{PU}(N)) \to H^2(\mathrm{PU}(N), \mathbb{Z}) \to 0.$$

Therefore the Cheeger-Simons characters of 1-cycles on $\mathrm{PU}(N)$, are in one-to-one correspondence with line bundles on $\mathrm{PU}(N)$. So we deduce that the map from line bundles on $\mathcal{P}$ to $\mathrm{SU}(N)$-actions on L that cover the $\mathrm{PU}(N)$ action on $\mathcal{P}$, is simply given by the map

$$H^2(\mathcal{P}, \mathbb{Z}) \ni c_1 \to r^* c_1 \in H^2(\mathrm{PU}(N), \mathbb{Z})$$

where r is the restriction map to any fiber.

One consequence of Proposition 2 is that any line bundle on $\mathcal{P}$ is necessarily the Nth root of a line bundle on the base.

Lemma 1. *If $\tilde{L}$ is a line bundle on $\mathcal{P}$ for a given Azumaya bundle then $\tilde{L}^{\otimes N}$ descends to a line bundle L over the base Z.*

Proof. It follows from the equivariance that the center $\mathbb{Z}_N$ of $\mathrm{SU}(N)$ acts on the fibre $\tilde{L}_q$, $q \in \mathcal{P}$, at each point as multiplication by Nth roots of unity. Thus, in the induced action on $\tilde{L}^{\otimes N}$ the center acts trivially, so $\tilde{L}^{\otimes N}$ has an induced $\mathrm{PU}(N)$-action over $\mathcal{P}$ and so descends to a bundle on the base. q.e.d.

Lemma 2. *Any line bundle over $\mathcal{P}$ has a connection with curvature which is the lift of the form $\frac{1}{N}\pi^*\omega_L$ from the base where ω_L is the curvature on the base of a connection on $L = \tilde{L}^{\otimes N}$.*

Proof. Following the discussion in the proof of Proposition 2 a given line bundle L carries a connection with curvature, ω, which is $\mathrm{PU}(N)$-invariant and vanishes on the fibres. Taking a smooth lift of any vector field, v, to a vector field v^* on $\mathcal{P}$ the form $i_{v^*}\omega$ is well-defined independent of the lift, and closed on each fibre. Since $H^1(\mathrm{PU}(N), \mathbb{R}) = \{0\}$, it follows that the connection may be further modified, by a smooth 1-form which vanishes on each fibre, so that the curvature is a basic and $\mathrm{PU}(N)$-invariant form, i.e., is the lift of a form from the base. Computing in any local trivialization of $\mathcal{P}$ gives the curvature in terms of the induced connection on the Nth power. q.e.d.

Remark 5. Given a line bundle L over a compact manifold M, the problem of finding an N-th root of L which is also a line bundle on M can be approached as follows. One can take the N-th root of the transition functions of L with respect to a good cover U_a, $g_{ab}^{1/N} : U_{ab} \to \mathrm{U}(1)$. On triple overlaps, this gives a cocycle

$$t_{abc} = g_{ab}^{1/N} g_{bc}^{1/N} g_{ca}^{1/N} : U_{abc} \to \mathbb{Z}_N \tag{7}$$

where $\mathbb{Z}_N = ker(s)$, s being the second homomorphism in the short exact sequence

$$\mathbb{Z}_N \to \mathrm{U}(1) \to \mathrm{U}(1) \tag{8}$$

which is given by $s(z) = z^N$. Then the obstruction to the existence of an N-th root for L is given by the connecting homomorphism in the corresponding long exact sequence in cohomology

$$\begin{aligned}\cdots \to H^1(M, \underline{\mathrm{U}(1)}) &\xrightarrow{\beta} H^2(Z, \mathbb{Z}_N) \\ &\to H^2(M, \underline{\mathrm{U}(1)}) \to H^2(M, \underline{\mathrm{U}(1)}) \to \cdots\end{aligned} \tag{9}$$

i.e., the obstruction class is $\beta(L) \in H^2(M, \mathbb{Z}_N)$. In fact, $\beta(L) = c_1(\mathcal{L}) \pmod N$. If L has an Nth root L_0 then all of the other Nth roots of L are of the form $L_0 \otimes R$, where R is a line bundle on Z such that $R^{\otimes N} = 1$. Hence the set of Nth roots of L is a $\mathbb{Z}_N$-affine space with associated vector space $H^1(M, \mathbb{Z}_N)$.

Applying this to the case $M = \mathcal{P}$ it follows that all of the line bundles which have Nth powers a given bundle L over the base are of the form $\tilde{L} \otimes R$, where $R^{\otimes N} = 1$, so form a $\mathbb{Z}_N$-affine space with associated vector space $H^1(\mathcal{P}, \mathbb{Z}_N)$.

Recall that a principal $\mathrm{PU}(N)$ bundle $\pi : \mathcal{P} \to M$ has an invariant, $t(\mathcal{P}) \in H^2(M, \mathbb{Z}_N)$, which measures the obstruction to lifting $\mathcal{P}$ to a principal $\mathrm{SU}(N)$ bundle. This obstruction is obtained via the connecting homomorphism of the exact sequence in cohomology associated to the short exact sequence of sheaves of groups on M,

$$1 \to \mathbb{Z}_N \to \underline{\mathrm{SU}(N)} \to \underline{\mathrm{PU}(N)} \to 1,$$

namely $t(\mathcal{P}) = \delta(\mathcal{P})$, where $\delta : H^1(M, \underline{\mathrm{PU}(N)}) \to H^2(M, \mathbb{Z}_N)$ and is the mod N analogue of the 2nd Stieffel-Whitney class. Then a theorem of Serre (cf. [**4**]) asserts that given any class $t \in H^2(M, \mathbb{Z}_N)$, there is a principal $\mathrm{PU}(mN)$ bundle $\pi' : \mathcal{Q} \to M$ (for some $m \in \mathbb{N}$) such that $t(\mathcal{Q}) \in H^2(M, \mathbb{Z}_{mN})$ maps to $t \in H^2(M, \mathbb{Z}_N)$ under the standard inclusion of the coefficient groups. Such a bundle $\mathcal{Q}$ is by no means unique.

In particular, given any line bundle L on M, by the theorem of Serre [**4**], discussed above we know that there is a principal $\mathrm{PU}(mN)$ bundle $\pi' : \mathcal{Q} \to M$ (for some $m \in \mathbb{N}$) such that $t(\mathcal{Q}) = \beta(L) \in H^2(M, \mathbb{Z}_N)$. Since $t(\mathcal{Q})$ and $\beta(L)$ are characteristic classes,

$$0 = \pi'^*(t(\mathcal{Q})) = t(\pi'^*(\mathcal{Q})) = \beta(\pi'^*(L)) \in H^2(\mathcal{Q}, \mathbb{Z}_N),$$

that is, there is a line bundle $\tilde{L}$ on $\mathcal{Q}$ which isomorphic to an Nth root of the lifted line bundle $\pi'^*(L)$ on $\mathcal{Q}$, i.e., $\tilde{L}^{\otimes N} \cong \pi'^*(L)$.

Note that the exact sequence

$$1 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}_N \to 1,$$

where the middle arrow is multiplication by N, determines the change of coefficients long exact sequence in cohomology, where $\mathfrak{d} : H^2(M, \mathbb{Z}_N) \to H^3(M, \mathbb{Z})$ is one of the connecting homomorphisms. Then $\mathfrak{d}(t(\mathcal{P})) \in H^3(M, \mathbb{Z})$ is equal to the Dixmier-Douady invariant, which measures the obstruction to lifting $\mathcal{P}$ to a principal $\mathrm{U}(N)$ bundle over M. Of course, this is a less stringent requirement.

The proof of Proposition 1 applies just as well to the line bundle $\tilde{L} \boxtimes \tilde{L}^{-1}$ over $\mathcal{P} \times \mathcal{P}$.

Lemma 3. *If $\tilde{L}$ is a line bundle over $\mathcal{P}$, the bundle $\tilde{L} \boxtimes \tilde{L}^{-1}$ descends from a neighborhood of the diagonal submanifold of $\mathcal{P} \times \mathcal{P}$ to a well-defined line bundle $\widehat{L}$ over a neighborhood of the diagonal in Z^2.*

In general a line bundle over $\mathcal{P}$ with its $\mathrm{SU}(N)$ action, and the corresponding $\mathrm{U}(N)$ action, represents a 'partial trivialization' of the Azumaya bundle. If the $\mathbb{Z}_N$ action on the fibres of $\tilde{L}$, arising from the centre of $\mathrm{SU}(N)$, is injective then in fact the circle bundle associated to $\tilde{L}$ is a lift of the principal $\mathrm{PU}(N)$-bundle, $\mathcal{P}$, to a principal $\mathrm{U}(N)$-bundle. If, at the other extreme, this $\mathbb{Z}_N$ action is trivial then $\tilde{L}$ is simply the lift of a line bundle from the base.

Proposition 3. *Any representing bundle for an Azumaya bundle is equivalent to $\tilde{L} \otimes \tilde{E}$ with the induced $\mathrm{SU}(N)$ action, where $\tilde{E}$ is the trivial bundle with standard $\mathrm{SU}(N)$ action and $\tilde{L}$ is a line bundle on $\mathcal{P}$ with its fibre-holonomy $\mathrm{SU}(N)$ action.*

Proof. Let $\tilde{V}$ be a representing bundle for the Azumaya bundle $\mathcal{A}$. By assumption $\tilde{V}$ is a bundle over $\mathcal{P}$. Let $\mathcal{F}$ be the frame bundle for $\tilde{V}$. This is the principal U(N)-bundle with fibre at a point $p \in \mathcal{P}$ the space of trivializations of $\tilde{V}_p$. Now, as part of the data of a representing bundle, we are given an identification of $\mathcal{A}$ with $\hom(\tilde{V})$ as a bundle over the base. Since a point of $\mathcal{P}$ is an identification of the fibres of $\mathcal{A}$ with $N \times N$ matrices, this data picks out a $U(1)$ subbundle $\mathcal{L} \subset \mathcal{F}$, consisting of the isomorphisms between $\tilde{V}_p$ and $\mathbb{C}^N$ which realize this identification at that point. Since the equivariant U(N) action on $\tilde{V}$ has center acting as scalars, $\mathcal{L}$ has an induced equivariant U(N) action coming from the equivariant U(N) action on $\tilde{V}$ and the standard U(N) action on the trivial bundle. If we let $\tilde{L}$ be the line bundle over $\mathcal{P}$ associated to $\mathcal{L}$ then it has a U(N) action and the restriction of this to SU(N) must be the SU(N) action. The frame bundle of $\tilde{L}^{-1} \otimes \tilde{V}$ has a natural U(N)-invariant section over $\mathcal{P}$, so $\tilde{V}$ is equivalent, as a representing bundle, to $\tilde{L} \otimes \tilde{E}$ where $\tilde{E}$ is the standard, trivial, representing bundle. q.e.d.

In view of this result we generalize projective bundles slightly by allowing twisting by line bundles over $\mathcal{P}$.

Definition 2. For any Azumaya bundle $\mathcal{A}$ and line bundle, $\tilde{L}$, over $\mathcal{P}$ an associated ($\tilde{L}-$)twisted projective bundle is a subbundle of $(\tilde{L} \otimes \mathbb{C}^N) \otimes \mathcal{H}$ which is invariant under the tensor product SU(N) action, arising from the SU(N) action on $\tilde{L}$ and the standard SU(N) action on $\mathbb{C}^N$ interpreted as covering the PU(N) action on $\mathcal{P}$.

Proposition 4. *Any choice of representing bundle, $\tilde{V} \equiv \tilde{L} \otimes \tilde{E}$, for an Azumaya bundle $\mathcal{A}$ over Z gives rise to a vector bundle $\widehat{\mathcal{A}}_{\tilde{L}}$ which is defined in a neighborhood of the diagonal of Z^2, extends $\mathcal{A} = \hom(V)$ from the diagonal, has the composition property* (5) *and lifts canonically to* $\operatorname{Hom}(\tilde{V})$ *over a neighborhood of the diagonal on $\mathcal{P} \times \mathcal{P}$; with its composition maps* (5) *it is isomorphic to $\widehat{\mathcal{A}} \otimes \widehat{L}$ where $\tilde{L}$ is given by Proposition* 3*.*

Proof. The proof of Proposition 1 may be used directly, since no use is made of the fact that the SU(N) action there is the standard one.
q.e.d.

Remark 6. The same argument also gives an extension $\operatorname{Hom}^{\mathcal{A},\tilde{L}}(E, F)$ of $\hom(E, F)$ for any two $\tilde{L}$-twisted projective bundles $\tilde{E}$ and $\tilde{F}$ for the same line bundle over $\mathcal{P}$.

Remark 7. Applying Proposition 4 to the 'trivial' case of an Azumaya bundle $\mathcal{A} = \hom(W)$ for a vector bundle W we recover recover $\widehat{\mathcal{A}}_W = \operatorname{Hom}^{\mathcal{A},\tilde{L}} = \operatorname{Hom}(W)$ in a neighborhood of the diagonal.

3. Trivialization and spin structures

Corresponding to (3), the principal $\mathrm{PU}(N)$-bundle $\mathcal{P}$ of local trivializations of the Azumaya bundle $\mathcal{A}$ may have a lift to a principal $\mathrm{SU}(N)$-bundle, or to a principal $\mathrm{U}(N)$-bundle,

(10)
$$\begin{tikzcd} \mathrm{SU}(N) \arrow[r] \arrow[d] & \mathcal{P}_{\mathrm{SU}(N)} \arrow[d] \\ \mathrm{PU}(N) \arrow[r] & \mathcal{P} \end{tikzcd} \quad \text{or} \quad \begin{tikzcd} \mathrm{U}(N) \arrow[r] \arrow[d] & \mathcal{P}_{\mathrm{U}(N)} \arrow[d] \\ \mathrm{PU}(N) \arrow[r] & \mathcal{P}. \end{tikzcd}$$

In either case the Dixmier-Douady invariant of $\mathcal{A}$ vanishes; conversely the vanishing of the invariant implies the existence of such a lifting to a $\mathrm{U}(pN)$-bundle for $\mathcal{A}\otimes\hom(G)$ for some bundle G (and so with rank a multiple of N.)

Since it is of primary concern below, consider the special case of the Clifford bundle. Namely, a choice of metric on Z defines the bundle of Clifford algebras, with fibre at $z\in Z$ the (real or complexified) Clifford algebra

(11)
$$\begin{gathered}\mathrm{Cl}_z(Z)=\left(\bigoplus_{k=0}^{\infty}(T_z^*Z)^k\right)/\langle\alpha\otimes\beta+\beta\otimes\alpha-2(\alpha,\beta)_g,\ \alpha,\beta\in T_z^*Z\rangle,\\ \mathbb{C}\mathrm{l}_z(Z)=\mathbb{C}\otimes\mathrm{Cl}_z(Z).\end{gathered}$$

If $\dim Z=2n$, this complexified algebra is isomorphic to the matrix algebra on $\mathbb{C}^{2^n}$. A local smooth choice of orthonormal basis over an open set $\Omega\subset Z$ identifies $T^*\Omega$ with $\Omega\times\mathbb{R}^{2n}$ and so identifies $\mathrm{Cl}(\Omega)$ with $\Omega\times\mathrm{Cl}(\mathbb{R}^{2n})$ as Azumaya bundles. Choosing a fixed identification of $\mathbb{C}\mathrm{l}(2n)$ with the algebra of complex $2^n\times 2^n$ matrices therefore gives a trivialization of $\mathbb{C}\mathrm{l}(Z)$, as an Azumaya bundle, over Ω. As noted in [**10**], its Dixmier-Douady invariant is $W_3(Z)$.

In particular the Clifford bundle is an associated bundle to the metric coframe bundle, the principal $\mathrm{SO}(2n)$-bundle $\mathcal{F}$, where the action of $\mathrm{SO}(2n)$ on the Euclidean Clifford algebra $\mathrm{Cl}(2n)$ is through the spin group. Thus, the spin group may be identified within the Clifford algebra as

(12) $$\mathrm{Spin}(2n)=\{v_1v_2\cdots v_{2k}\in\mathrm{Cl}(2n);v_i\in\mathbb{R}^{2n},\ |v_i|=1\}.$$

The non-trivial double covering of $\mathrm{SO}(2n)$ comes through the mapping of v to the reflection $R(v)\in\mathrm{O}(2n)$ in the plane orthogonal to v

(13) $$p:\mathrm{Spin}(2n)\ni a=v_1\cdots v_{2k}\longmapsto R(v_1)\cdots R(v_{2k})=R\in\mathrm{SO}(2n).$$

Thus $\mathcal{P}$ may be identified with the bundle associated to $\mathcal{F}$ by the action of $\mathrm{SO}(2n)$ on $\mathbb{C}\mathrm{l}(2n)$ (or in the real case $\mathrm{Cl}(2n)$) where R in (13) acts by conjugation by a

$$\mathrm{Cl}(2n) \ni b \longmapsto aba^{-1} \in \mathrm{Cl}(2n). \tag{14}$$

We therefore have a map of principal bundles

$$\mathcal{F} \longrightarrow \mathcal{P}. \tag{15}$$

Recall that the projective spin bundle on $\mathcal{P}$ is just the bundle associated to the natural action of $\mathrm{Cl}(n)$ on itself; it can therefore be identified with the trivial bundle over $\mathcal{P}$ with an equivariant $\mathrm{SU}(N)$ action, where $N = 2^n$.

Now, a spin structure on Z, corresponds to an extension, $\mathcal{F}_S$, of the coframe bundle to a Spin bundle,

$$\begin{tikzcd} \mathrm{Spin}(2n) \arrow[r] \arrow[d, "p"] & \mathcal{F}_S \arrow[d] \\ \mathrm{SO}(2n) \arrow[r] & \mathcal{F}. \end{tikzcd} \tag{16}$$

Since $\mathrm{Spin}(2n) \subset \mathrm{SU}(N)$, where $\mathrm{SU}(N) \subset \mathbb{C}\mathrm{l}(2n)$, this in turn gives rise to a lift of $\mathcal{P}$ to a principal $\mathrm{SU}(N)$ bundle:

$$\begin{tikzcd} \mathrm{SU}(N) \arrow[rr] & & \mathcal{P}_{\mathrm{SU}(N)} \arrow[dd] \\ \mathrm{Spin}(2n) \arrow[u] \arrow[r] \arrow[d, "p"] & \mathcal{F}_S \arrow[d] \arrow[ur, dotted] & \\ \mathrm{SO}(2n) \arrow[r] & \mathcal{F} \arrow[r] & \mathcal{P}. \end{tikzcd} \tag{17}$$

Thus the projective bundle naturally associated to the Clifford bundle can reasonably be called the projective spin bundle since a spin structure on the manifold gives a lift of $\tilde{E} \otimes M$, where M is the $\mathbb{Z}_2$ bundle given by the spin structure, to the usual spin bundle.

As in the standard case, the Levi-Civita connection induces a natural, $\mathrm{SU}(N)$ equivariant, connection on the projective spin bundle over $\mathcal{P}$. We use this below to define the projective spin Dirac operator; a choice of spin structure, when there is one, identifies it with the spin Dirac operator.

Note that similar remarks apply to a $\mathrm{spin}_{\mathbb{C}}$ structure on the manifold Z. The model group

$$\begin{aligned} (18)\quad & \mathrm{Spin}_{\mathbb{C}}(2n) \\ &= \{cv_1v_2\cdots v_{2k} \in \mathrm{Cl}(2n); v_i \in \mathbb{R}^{2n},\ |v_i| = 1, c \in \mathbb{C},\ |c| = 1\} \\ &\qquad\qquad = (\mathrm{Spin} \times \mathrm{U}(1))/\pm, \end{aligned}$$

is a central extension of $\mathrm{SO}(2n)$,

$$\mathrm{U}(1) \longrightarrow \mathrm{Spin}_{\mathbb{C}}(2n) \longrightarrow \mathrm{SO}(2n) \tag{19}$$

where the quotient map is consistent with the covering of $\mathrm{SO}(2n)$ by $\mathrm{Spin}(2n)$.

Thus a $\mathrm{spin}_{\mathbb{C}}$ structure is an extension of the coframe bundle to a principal $\mathrm{Spin}_{\mathbb{C}}(2n)$-bundle;

$$\begin{tikzcd}
\mathrm{U}(1) \arrow[r] \arrow[d] & L \\
\mathrm{Spin}_{\mathbb{C}}(2n) \arrow[r] \arrow[d] & \mathcal{F}_L \arrow[d] \\
\mathrm{SO}(2n) \arrow[r] & \mathcal{F}.
\end{tikzcd} \tag{20}$$

where $\mathcal{F}_L$, the $\mathrm{Spin}_{\mathbb{C}}(2n)$ bundle, may be viewed as a circle bundle over $\mathcal{F}$. Since $\mathrm{Spin}_{\mathbb{C}}(2n) \hookrightarrow \mathrm{U}(N)$ (but is not a subgroup of $\mathrm{SU}(N)$) this gives a diagram similar to (17) but lifting to a principal $\mathrm{U}(N)$ bundle

$$\begin{tikzcd}
\mathrm{U}(N) \arrow[rr] & & \mathcal{P}_{\mathrm{U}(N)} \arrow[dd] \\
\mathrm{Spin}_{\mathbb{C}}(2n) \arrow[u] \arrow[r] \arrow[d, "p"] & \mathcal{F}_L \arrow[ur, dotted] \arrow[d] & \\
\mathrm{SO}(2n) \arrow[r] & \mathcal{F} \arrow[r] & \mathcal{P}.
\end{tikzcd} \tag{21}$$

In this case the $\mathrm{spin}_{\mathbb{C}}$ bundle over Z is the lift of $S \otimes L$ from $\mathcal{P}$ to $\mathcal{P}_{\mathrm{U}(N)}$.

Note that the existence of a spin structure on Z is equivalent to the condition $w_2 = 0$. The Clifford bundle is then the homomorphism bundle of the spinor bundle, so the existence of a spin structure implies the vanishing of the Dixmier-Douady invariant of the Clifford bundle (which is W_3, the Bockstein of w_2); the vanishing of W_3 is precisely equivalent to the existence of a $\mathrm{spin}_{\mathbb{C}}$ structure (without any necessity for stabilization).

In the general case, even when $W_3 \neq 0$ and there is no $\mathrm{spin}_{\mathbb{C}}$ structure, we shall show below that we can still introduce the notion of a 'projective $\mathrm{spin}_{\mathbb{C}}$ Dirac operator' starting from the following notion.

Definition 3. On any even-dimensional, oriented manifold a projective $\mathrm{spin}_{\mathbb{C}}$ structure is a choice of representing bundle, in the sense of Definition 1, for the complexified Clifford bundle.

Thus, by Proposition 3 such a representing bundle is always equivalent to, and hence can be replaced by, $\tilde{S} \otimes \tilde{L}$ where $\tilde{L}$ is a line bundle over the bundle of trivializations $\mathcal{P}_{\mathbb{C}\mathrm{l}}$ of the Clifford bundle and $\tilde{S}$ is the projective spin bundle. As remarked above, this is consistent with the standard case in which there is a $\mathrm{spin}_{\mathbb{C}}$ structure and $\tilde{V}$ then descends to a bundle on the base.

Remark 8. Any line bundle over $\mathcal{P}_{\mathbb{C}\mathrm{l}}$ is necessarily a square root of a line bundle from the base. This follows by restricting the line bundle to the frame bundle, as a subbundle of $\mathcal{P}_{\mathbb{C}\mathrm{l}}$, and the $\mathrm{U}(N)$, $N = 2^n$, action to $\mathrm{Spin}(2n)$ showing that the centre acts through the subgroup $\mathbb{Z}_2 \subset \mathbb{Z}_N$.

4. Smoothing operators

For two vector bundles E and F the space of smoothing operators $\Psi^{-\infty}(Z; E, F)$ between sections of E and sections of F may be identified with the corresponding space of kernels

$$\Psi^{-\infty}(Z; E, F) = \mathcal{C}^\infty(Z^2; \mathrm{Hom}(E, F) \otimes \pi_R^*\Omega) \tag{22}$$

where the section of the density bundle allows invariant integration. Thus, such kernels define linear maps $\mathcal{C}^\infty(Z; E) \longrightarrow \mathcal{C}^\infty(Z; F)$ through

$$Au(z) = \int_Z A(z, z')u(z'). \tag{23}$$

Operator composition induces a product

$$\begin{gathered}\Psi^{-\infty}(Z; F, G) \circ \Psi^{-\infty}(Z; E, F) \subset \Psi^{-\infty}(Z; E, G), \\ A \circ B(z, z'') = \int_Z A(z, z')B(z', z'')\end{gathered} \tag{24}$$

using the composition law (4). The right density factor in A is used in (24) to carry out the integral invariantly.

Given the extensions in Proposition 1 and Proposition 4 of the homomorphism bundles it is possible to define the linear space of smoothing operators with kernels supported in N_ϵ for any pair E, F of projective bundles (or twisted projective bundles) associated to a fixed Azumaya bundle (and twisting) as

$$\Psi_\epsilon^{-\infty}(Z; E, F) = \mathcal{C}_{\mathrm{c}}^\infty(N_\epsilon; \mathrm{Hom}^{\mathcal{A},\tilde{L}}(E, F) \otimes \pi_R^*\Omega) \tag{25}$$

where in case E and F are projective bundles, without twisting, $\tilde{L}$ is trivial so is dropped from the notation. Note that the projective and possibly twisted nature of E and F is implicit in the notation. Although there is no action analogous to (23) the composition law (4) allows (24) to be extended directly to define

$$\Psi_{\epsilon/2}^{-\infty}(Z; F, G) \circ \Psi_{\epsilon/2}^{-\infty}(Z; E, F) \subset \Psi_\epsilon^{-\infty}(Z; E, G) \tag{26}$$

in the case of three projective bundles associated to the fixed $\mathcal{A}$. For sufficiently small supports this product is associative

$$\begin{aligned}&(A\circ B)\circ C = A\circ(B\circ C)\\ &\qquad \text{if } A\in\Psi^{-\infty}_{\epsilon/4}(Z;G,H),\ B\in\Psi^{-\infty}_{\epsilon/4}(Z;F,G),\ C\in\Psi^{-\infty}_{\epsilon/4}(Z;E,F).\end{aligned}$$

The trace functional extends naturally to these spaces

$$\mathrm{Tr}:\Psi^{-\infty}_{\epsilon}(Z;E)=\int_Z \operatorname{tr} A(z,z) \tag{27}$$

and vanishes on appropriate commutators

$$\mathrm{Tr}(AB-BA)=0 \text{ if } A\in\Psi^{-\infty}_{\epsilon/2}(Z;F,E),\ B\in\Psi^{-\infty}_{\epsilon/2}(Z;E,F) \tag{28}$$

as follows from Fubini's theorem.

5. Pseudodifferential operators

Just as the existence of the bundle $\mathrm{Hom}^{\mathcal{A},\tilde{L}}(E,F)$ over the neighborhood N_ϵ of the diagonal allows smoothing operators to be defined, it also allows arbitrary pseudodifferential operators, with kernels supported in N_ϵ to be defined as the space of kernels

$$\Psi^m_\epsilon(Z;E,F)=I^m_{\mathrm{c}}(N_\epsilon;\mathrm{Diag})\otimes_{\mathcal{C}^\infty_{\mathrm{c}}(N_\epsilon)}\mathcal{C}^\infty_{\mathrm{c}}(N_\epsilon;\mathrm{Hom}^{\mathcal{A},\tilde{L}}(E,F)). \tag{29}$$

Here, E, F are either $\tilde{L}$-twisted projective bundles associated to some Azumaya bundle $\mathcal{A}$. These are just the conormal sections of $\mathrm{Hom}^{\mathcal{A},\tilde{L}}(E,F)\otimes\pi_R^*\Omega$ with support in N_ϵ. Notice that for any small $\delta<\epsilon$,

$$\Psi^m_\delta(Z;E,F)+\Psi^{-\infty}_\epsilon(Z;E,F)=\Psi^m_\epsilon(Z;E,F). \tag{30}$$

The singularities of these kernels are unrestricted by the support condition so there are the usual short exact sequences

(31)
$$\begin{gathered}\Psi^{m-1}_\epsilon(Z;E,F)\longrightarrow\Psi^m_\epsilon(Z;E,F)\xrightarrow{\sigma_m}\mathcal{C}^\infty(S^*Z;\hom(E,F)\otimes N_m)\\ \Psi^{-\infty}_\epsilon(Z;E,F)\longrightarrow\Psi^m_\epsilon(Z;E,F)\xrightarrow{\sigma}\rho^{-m}\mathcal{C}^\infty(S^*Z;\hom(E,F))[[\rho]].\end{gathered}$$

In the first case N_m is the bundle over S^*Z of the smooth functions on $T^*Z\setminus 0$ which are homogeneous of degree m; this sequence is completely natural and independent of choices. In the second sequence $\rho\in\mathcal{C}^\infty(\overline{T^*Z})$ is a defining function for the boundary and the image space represents Taylor series at the boundary, with an overall factor of ρ^{-m}; this sequence depends on choices of a metric and connection to give a quantization map.

The product for pseudodifferential operators extends by continuity (using the larger spaces of symbols with bounds, rather than the classical symbols implicitly used above) from the product for smoothing operators and leads to an extension of (26)

$$\tag{32} \begin{gathered} \Psi^m_{\epsilon/2}(Z;F,G)\circ\Psi^{m'}_{\epsilon/2}(Z;E,F)\subset\Psi^{m+m'}_{\epsilon}(Z;E,G),\\ \sigma_{m+m'}(A\circ B)=\sigma_{m'}(A)\sigma_m(B). \end{gathered}$$

The induced product on the image space given by the second short exact sequence in (31) is a star product as usual.

The approximability of general pseudodifferential operators by smoothing operators, in the weaker topology of symbols with bounds, also shows, as in the standard case, that (28) extends to

$$\tag{33} \operatorname{Tr}([A,B])=0 \text{ if } A\in\Psi^m_{\epsilon/2}(Z;F,E),\ B\in\Psi^{-\infty}_{\epsilon/2}(Z;E,F)$$

for any m.

Now, the standard *symbolic* constructions of the theory of pseudodifferential operators carry over directly since these are all concerned with the diagonal singularity and the symbol map.

Theorem 1. *For any two projective bundles associated to the same Azumaya bundle (or twisted projective bundles associated to the same Azumaya bundle and the same line bundle over $\mathcal{P}$), if $A\in\Psi^m_{\epsilon/2}(Z;E,F)$ is elliptic, in the sense that $\sigma_m(A)$ is invertible (pointwise), then there exists $B\in\Psi^{-m}_{\epsilon/2}(Z;F,E)$ such that*

$$\tag{34} B\circ A=\operatorname{Id}-E_R,\ A\circ B=\operatorname{Id}-E_L,\ E_R\in\Psi^{-\infty}_{\epsilon}(Z;E),\ E_L\in\Psi^{-\infty}_{\epsilon}(Z;F)$$

and any two such choices B', B satisfy $B'-B\in\Psi^{-\infty}_{\epsilon/2}(Z;E,F)$.

Proof. Now absolutely standard. q.e.d.

If B' and B are two such parametrices it follows that $B_t=(1-t)B'+tB$, $t\in[0,1]$, is a smooth curve of parametrices. Furthermore

$$\tag{35} \frac{d}{dt}[A,B_t]=[A,(B-B')]$$

so, by (33), it follows that for any two parametrices

$$\tag{36} \operatorname{Tr}([A,B'])=\operatorname{Tr}([A,B])$$

since $B'-B$ is smoothing.

Definition 4. For an elliptic pseudodifferential operator $A\in\Psi^m_{\epsilon/2}(Z;E,F)$ acting between projective bundles associated to a fixed Azumaya bundle, or more generally between twisted projective bundles corresponding to the same twisting line bundle over $\mathcal{P}$, we define

$$\tag{37} \operatorname{ind}_a(A)=\operatorname{Tr}(AB-\operatorname{Id}_F)-\operatorname{Tr}(BA-\operatorname{Id}_E)$$

for any parametrix as in Theorem 1.

6. Homotopy invariance

Proposition 5. *The index* (37) *is constant on a* 1*-parameter family of elliptic operators.*

Remark 9. Given the rationality proved in the next section this follows easily. Here we use the homotopy invariance to prove the rationality!

Proof. For a smooth family $A_t \in \mathcal{C}^\infty([0,1]; \Psi^m_{\epsilon/2}(Z; E, F))$ of elliptic operators as discussed above, it follows as in the standard case that there is a smooth family of parametrices, $B_t \in \mathcal{C}^\infty([0,1]; \Psi^{-m}_{\epsilon/2}(Z; E, F))$. Thus the index, defined by (37) is itself smooth, since $[B_t, A_t]$ is a smooth family of smoothing operators. To prove directly that this function is constant we use the residue trace of Wodzicki, see [**16**], with an improvement to the definition due to Guillemin [**5**] and also the trace defect formula from [**13**].

For a classical operator A of integral order, m, in the usual calculus the residue trace is defined by 'ζ-regularization' (following ideas of Seeley) using the entire family of complex powers of a fixed positive (so self-adjoint) elliptic operator of order 1 :

$$\operatorname{Tr}_{\mathrm{R}}(A) = \lim_{z \to 0} z \operatorname{Tr}(AD^z) \tag{38}$$

where $\operatorname{Tr}(AD^z)$ is known to be meromorphic with at most simple poles at $z = -k - \dim Z + \{0, 1, 2, \dots\}$. One of Guillemin's innovations was to show that the same functional results by replacing D^z by any entire family $D(z)$ of pseudodifferential operators of complex order z which is elliptic and has $D(0) = \operatorname{Id}$.

One way to construct such a family, which is useful below, is to choose a generalized Laplacian on the bundle in question, which is to say a second order self-adjoint differential operator, L, with symbol $|\xi|^2 \operatorname{Id}$, the metric length function, and to construct its heat kernel, $\exp(-tL)$. This is a well-defined (locally integrable) section of the homomorphism bundle on $[0, \infty)_t \times Z^2$ which is singular only at $\{t = 0\} \times \operatorname{Diag}(Z)$ and vanishes with all derivatives at $t = 0$ away from the diagonal. If $L = D^2$ is strictly positive the heat kernel decays exponentially as $t \to \infty$ and the complex powers of L are given by the Mellin transform

$$L^z = D^{2z} = \frac{1}{\Gamma(z)} \int_0^\infty t^{-z-1} \exp(-tL) dt, \tag{39}$$

where the integral converges for $\operatorname{Re} z << 0$ and extends meromorphically to the whole of the complex plane. The fact that $D^0 = \operatorname{Id}$ arises from the residue of the integral at $z = 0$, so directly from the fact that $\exp(-tL) = \operatorname{Id}$ at $t = 0$. It follows that if $\chi \in \mathcal{C}^\infty_{\mathrm{c}}([0, \infty) \times Z^2)$ and

$\chi \equiv 1$, in the sense of Taylor series, at $\{0\} \times \mathrm{Diag}$ then

$$D(2z) = \frac{1}{\Gamma(z)} \int_0^\infty t^{-z-1} H(t) dt, \quad H(t) = \chi \exp(-tL) \tag{40}$$

is an entire family as required for Guillemin's argument, that is D is elliptic and $D(0) = \mathrm{Id}$. Since the construction of the singularity of the heat kernel for such a differential operator is completely symbolic (see for instance Chapter 5 of [**12**]), quite analogous to the construction of a parametrix for an elliptic operator, it can be carried out in precisely the same manner in the projective case, so giving a family of the desired type via (40).

Alternatively, for any projective vector bundle E, such a family can be constructed using an explicit linear quantization map, with kernels supported arbitrarily close to the diagonal

$$D_E(z) \in \Psi^z_{\epsilon/4}(Z; E). \tag{41}$$

Thus we may define the residue trace and prove its basic properties as in the standard case; in particular it vanishes on all operators of sufficiently low order. It is also a trace functional

$$\mathrm{Tr_R}([A, B]) = 0,\ A \in \Psi^m_{\epsilon/4}(Z; E, F),\ B \in \Psi^{m'}_{\epsilon/4}(Z; F, E). \tag{42}$$

The additional result from [**13**], see also [**14**], that we use here concerns the regularized trace. This is defined to be

$$\overline{\mathrm{Tr}}_{D_E}(A) = \lim_{z \to 0} \left(\mathrm{Tr}(AD_E(z)) - \frac{1}{z} \mathrm{Tr_R}(A) \right). \tag{43}$$

For general A it does depend on the regularizing family, but for smoothing operators it reduces to the trace. Therefore

$$\mathrm{ind}_a(A) = \overline{\mathrm{Tr}}_D([A, B]) = \overline{\mathrm{Tr}}_{D_F}(AB - \mathrm{Id}_F) - \overline{\mathrm{Tr}}_{D_E}(BA - \mathrm{Id}_E) \tag{44}$$

for an elliptic operator $A \in \Psi^m_{\epsilon/4}(Z; E, F)$ and $B \in \Psi^{m'}_{\epsilon/4}(Z; F, E)$ a parametrix for A. It is not a trace function but rather the 'trace defect' satisfies

$$\overline{\mathrm{Tr}}_D([A, B]) = \mathrm{Tr_R}(B \delta_D A) \tag{45}$$

where $\delta_D : \Psi^\bullet_\epsilon(Z; E, F) \to \Psi^\bullet_\epsilon(Z; E, F)$ is defined as

$$\delta_D A = \frac{d}{dz}\Big|_{z=0} D_F(z) A D_E(-z)$$

as in [**11**]. Then one computes,

$$
\begin{aligned}
\operatorname{ind}_a(A) &= \lim_{z\to 0}\left(\operatorname{Tr}(ABD_F(z)) - \operatorname{Tr}(BAD_E(z))\right) \\
&\quad - \overline{\operatorname{Tr}}_{D_F}(\operatorname{Id}_F) + \overline{\operatorname{Tr}}_{D_E}(\operatorname{Id}_E) \\
&= \lim_{z\to 0}\left(\operatorname{Tr}(BD_F(z)A) - \operatorname{Tr}(BAD_E(z))\right) \\
&\quad - \overline{\operatorname{Tr}}_{D_F}(\operatorname{Id}_F) + \overline{\operatorname{Tr}}_{D_E}(\operatorname{Id}_E) \\
&= \lim_{z\to 0}\left(\operatorname{Tr}(B(D_F(z)AD_E(-z) - A)D_E(z))\right) \\
&\quad - \overline{\operatorname{Tr}}_{D_F}(\operatorname{Id}_F) + \overline{\operatorname{Tr}}_{D_E}(\operatorname{Id}_E) \\
&= \operatorname{Tr}_{\mathrm{R}}(a^{-1}\delta_D a) - \overline{\operatorname{Tr}}_{D_F}(\operatorname{Id}_F) + \overline{\operatorname{Tr}}_{D_E}(\operatorname{Id}_E)
\end{aligned} \tag{46}
$$

where a is the image of A in the full symbol space and we observe that $D_E(z)D_E(-z) = \operatorname{Id}_E + O(z^2)$. Now δ_D also satisfies

$$
\operatorname{Tr}_{\mathrm{R}}(\delta_D a) = 0 \ \forall\ a, \tag{47}
$$

and when $E = F$, it is a derivation acting on the full symbol algebra in (31).

From these formulæ the homotopy invariance of the index in the projective case follows. Namely

$$
\begin{aligned}
\frac{d}{dt}\operatorname{ind}_a(A_t) &= \operatorname{Tr}_{\mathrm{R}}(a_t^{-1}\delta_D\dot{a}_t) + \operatorname{Tr}_{\mathrm{R}}((\frac{d}{dt}a_t^{-1})\delta_D a_t,) \\
&= -\operatorname{Tr}_{\mathrm{R}}(\dot{a}_t\delta_D a_t^{-1}) - \operatorname{Tr}_{\mathrm{R}}(a_t^{-1}\dot{a}_t a_t^{-1}\delta_D a_t) = 0.
\end{aligned} \tag{48}
$$

Here, a_t is the image of A_t in the full symbol space in which the image of B_t is a_t^{-1} and (47) has been used. q.e.d.

Remark 10. A similar argument also proves the multiplicativity of the index. Thus if A_i for $i = 1, 2$ are two elliptic projective operators with the image bundle of the first being the same as the domain bundle of the second, they can be composed if their supports are sufficiently small. Let B_i be corresponding parametrices, again with very small supports. Then B_1B_2 is a parametrix for A_2A_1 and the index of the product is given by (45) in terms of the 'full symbols' a_i of the A_i

$$
\begin{aligned}
\operatorname{ind}_a(A_2A_1) &= \overline{\operatorname{Tr}}_D([A_2A_1, B_1B_2]) = \operatorname{Tr}_{\mathrm{R}}(a_1^{-1}a_2^{-1}\delta_D(a_2a_1)) \\
&= \operatorname{Tr}_{\mathrm{R}}(a_1^{-1}\delta_D a_1) + \operatorname{Tr}_{\mathrm{R}}(a_2^{-1}\delta_D a_2) \\
&= \operatorname{ind}_a(A_1) + \operatorname{ind}_a(A_2).
\end{aligned} \tag{49}
$$

Remark 11. Another consequence of the homotopy invariance of the index is, as noted after the definition, that it is necessarily real. We do not use the reality in the proof of the index formula below, from which it follows that the index is rational, so we only sketch the argument.

First observe that there is an elliptic operator of any order, on any projective bundle, of index 0. Namely $D(m)$, discussed above, has this property, since it commutes with the regularizing family $D(z)$ in the symbol algebra, so (when $E = F$) the index vanishes from (46). Thus, using the multiplicativity, we need only consider the case of operators of order 0.

Next note that, using inner products on the projective bundles, $\operatorname{ind}_{\mathrm{a}}(P^*) = -\operatorname{ind}(P)$ for any elliptic operator P. To see this, consider the operator on the direct sum of the bundles

$$\tilde{P} = \begin{pmatrix} 0 & P^* \\ -P & 0 \end{pmatrix}. \tag{50}$$

If Q is a parametrix for P then

$$\begin{pmatrix} 0 & -Q \\ Q^* & 0 \end{pmatrix}.$$

is a parametrix for (50). Inserting this into the definition of the index it follows directly that the index of $\tilde{P}$ is $\operatorname{ind}_{\mathrm{a}}(P) + \operatorname{ind}_{\mathrm{a}}(P^*)$.

Now, $\tilde{P}$ can be imbedded in the elliptic family

$$\begin{pmatrix} \sin(\theta)\operatorname{Id}_E & \cos(\theta)P^* \\ -\cos(\theta)P & \sin(\theta)\operatorname{Id}_F \end{pmatrix}. \tag{51}$$

From the homotopy invariance it follows that the index is zero, thus indeed $\operatorname{ind}_{\mathrm{a}}(P) = -\operatorname{ind}_{\mathrm{a}}(P^*)$. However, simply taking the complex conjugate of (37) it then follows that

$$\overline{\operatorname{ind}_{\mathrm{a}}(P)} = -\operatorname{ind}_{\mathrm{a}}(P^*) = \operatorname{ind}_{\mathrm{a}}(P) \text{ is real.} \tag{52}$$

7. Projective Dirac operators

The space of differential operators 'acting between' two projective bundles associated to the same Azumaya algebra is well defined, since these are precisely the pseudodifferential operators with kernels with supports contained in the diagonal; we denote by $\operatorname{Diff}^k(Z; E, F)$ the space of these operators of order at most k.

Of particular interest is that, in this projective sense, there is a 'spin Dirac operator' on every oriented even-dimensional compact manifold. As discussed in Section 3 above, the projective bundle associated to the spin representation is the projective spin bundle of Z, which we denote by S; if Z is oriented it splits globally as the direct sum of two projective bundles $S^{\pm}$. There are natural connections on $\mathbb{C}\mathrm{l}(Z)$ and $S^{\pm}$ arising from the Levi-Civita connection on T^*Z. As discussed in Proposition 1, the homomorphism bundle of S, which can be identified with $\mathbb{C}\mathrm{l}(Z)$, has an extension to $\widehat{\mathbb{C}\mathrm{l}}(Z)$ in a neighborhood of the diagonal, and this extended bundle also has an induced connection. The projective spin Dirac operator may then be identified with the distribution

$$\eth = \mathrm{cl} \cdot \nabla_L(\kappa_{\mathrm{Id}}),\ \ \kappa_{\mathrm{Id}} = \delta(z - z')\operatorname{Id}_S\,. \tag{53}$$

Here κ_{Id} is the kernel of the identity operator in $\mathrm{Diff}^*(Z;S)$ and ∇_L is the connection restricted to the left variables with cl the contraction given by the Clifford action of T^*Z on the left. As in the usual case, $\eth$ is elliptic and odd with respect to the $\mathbb{Z}_2$ grading of S and locally the choice of a spin structure identifies this projective spin Dirac operator with the usual spin Dirac operator.

More generally we can consider projective twists of this projective spin Dirac operator. If E is any unitary projective vector bundle over Z, associated to an Azumaya bundle $\mathcal{A}$ and equipped with a Hermitian connection then $S \otimes E$ is a projective bundle associated to $\mathbb{C}\mathrm{l}(Z) \otimes \mathcal{A}$ and it is a Clifford module in the sense that

$$\mathbb{C}\mathrm{l}(Z) \subset \hom(S \otimes E).$$

The direct extension of (53), using the tensor product connection, gives an element $\eth_E \in \mathrm{Diff}^1(Z; S \otimes E)$ which is again $\mathbb{Z}_2$ graded. In the special case that $S \otimes E$ is a bundle a related construction is given by M. Murray and M. Singer [**15**].

The relation between the index of twisted, projective spin Dirac operators (or more generally, projective elliptic operators) and the distribution index of transversally elliptic operators, will be discussed in a subsequent paper.

Theorem 2. *The positive part,* $\eth_E^+ \in \mathrm{Diff}^1(Z; S^+ \otimes E, S^- \otimes E)$ *of the projective spin Dirac operator twisted by a unitary projective vector bundle* E*, has index*

$$\mathrm{ind}_a(\eth_E^+) = \int_Z \widehat{A}(Z) \wedge \mathrm{Ch}_{\mathcal{A}}(E) \tag{54}$$

where $\mathrm{Ch}_{\mathcal{A}} : K^0(Z;\mathcal{A}) \longrightarrow H^{\mathrm{even}}(Z;\mathbb{Q})$ *is the Chern character in twisted K-theory.*

Proof. The proof via the local index formula, see [**2**] and also [**12**], carries over to the present case. As discussed in section 6, the truncated heat kernel $H(t)$, formally representing $\exp(-t\eth_E^2)$, near $\mathrm{Diag}_Z \times \{t = 0\}$, is well-defined as a smooth kernel on $Z^2 \times (0, \infty)$, with values in $\mathrm{Hom}^{\mathbb{C}\mathrm{l}\,\otimes\,\mathcal{A}}\,(S \otimes E) \otimes \Omega_R$, modulo an element of $\dot{\mathcal{C}}^\infty(Z^2 \times [0,\infty); \mathrm{Hom}^{\mathbb{C}\mathrm{l}\otimes\mathcal{A}}(S \otimes E) \otimes \Omega_R)$; that is vanishing to all orders at $t = 0$. Then we claim that the analogue of the McKean-Singer formula holds,

$$\mathrm{ind}_a(\eth_E^+) = \lim_{t \downarrow 0} \mathrm{STr}(H(t)) \tag{55}$$

where STr is the supertrace, the difference of the traces on $S^+ \otimes E$ and $S^- \otimes E$. The local index formula, as a result of rescaling, asserts the existence of this limit and its evaluation (54).

In the standard case the McKean-Singer formula (55), for the actual heat kernel, follows by comparison with the limit as $t \to \infty$, which explicitly gives the index. Indeed then the function $\mathrm{STr}(\exp(-t\eth_E^2))$ is constant in t. In the present case the index is defined directly through (1) so the argument must be modified. If $H^{\pm}(t)$ are the approximate heat kernels of $\eth_E^-\eth_E^+$ and $\eth_E^-\eth_E^+$ respectively, then both approach the identity as $t \downarrow 0$. Thus for smoothing operators $K^{\pm}$ on the appropriate bundles, $H^{\pm}(t)K^{\pm} \longrightarrow K^{\pm}$ as smoothing operators as $t \downarrow 0$. Thus, from the continuity of the trace on smoothing operators, the index can be rewritten

$$\mathrm{ind_a}(\eth_E^+) = \lim_{t\downarrow 0} \mathrm{Tr}\left((\eth_E^+ B - \mathrm{Id}_F)H^-(t)\right) - \lim_{t\downarrow 0} \mathrm{Tr}\left((B\eth_E^+ - \mathrm{Id}_E)H^+(t)\right)$$

where B is a parametrix for $\eth_E^+$.

For $t > 0$ these approximate heat kernels are smoothing, so the terms can be separated showing that

$$\begin{aligned}(56)\quad & \mathrm{ind_a}(\eth_E^+) \\ & = \lim_{t\downarrow 0} \mathrm{Tr}\left(H^+(t) - H^-(t)\right) + \lim_{t\downarrow 0} \mathrm{Tr}_E\left(B(\eth^+ H^+(t) - H^-(t)\eth_E^+)\right) \\ & = \lim_{t\downarrow 0} \mathrm{STr}(H(t)).\end{aligned}$$

Here we use the fact that the difference $\eth^+ H^+(t) - H^-(t)\eth_E^+$ is, again by the (formal) uniqueness of solutions of the heat equation, a smoothing operator which vanishes rapidly as $t \downarrow 0$. This term therefore makes no contribution to the index and we recover (55) and hence the local index formula for projective Dirac operators. q.e.d.

Let $\mathcal{P}$ be the principal $\mathrm{PU}(N)$ bundle associated to $\mathcal{A}$ and $\mathcal{P}'$ be the principal $\mathrm{PU}(N')$ bundle associated to $\mathbb{Cl}(Z)$, cf. Section 3. Twisting by $\mathcal{P}$ and $\mathcal{P}'$-twisting line bundles $\tilde{L}$ and $\tilde{L}'$ respectively, does not affect the local discussion, only the final formula. Thus if $\tilde{S}_{\tilde{L}'} = \tilde{S} \otimes \tilde{L}'$ is a projective $\mathrm{spin}_{\mathbb{C}}$ bundle in the sense of Definition 3 and $\tilde{E}_{\tilde{L}} = \tilde{E} \otimes \tilde{L}$ is a projective vector bundle we may define the twisted projective $\mathrm{spin}_{\mathbb{C}}$ Dirac operator on it by choice of an $\mathrm{SU}(N')$-invariant connection on the twisting bundle $\tilde{L}'$ and $\mathrm{SU}(N)$-invariant connection on the twisting bundle $\tilde{L}$, the Levi-Civita connection on $\tilde{S}$ and an $\mathrm{SU}(N)$-invariant connection on $\tilde{E}$. As usual we think of these bundles, $S_{L'}, E_L$ as twisted projective bundles over the manifold, although they are in fact bundles over $\mathcal{P}'$ and $\mathcal{P}$ respectively.

Theorem 3. *The positive part,* $\eth_{L',E_L}^+ \in \mathrm{Diff}^1(Z; S_{L'}^+ \otimes E_L, S_{L'}^- \otimes E_L)$ *of the projective* $\mathrm{spin}_{\mathbb{C}}$ *Dirac operator corresponding to a general projective* $\mathrm{spin}_{\mathbb{C}}$ *structure and twisted by a unitary projective vector bundle* E *has index*

$$\begin{aligned}(57)\quad & \mathrm{ind}_a(\eth_{L',E_L}^+) \\ & = \int_Z \widehat{A}(Z) \wedge \exp\left(\frac{1}{2}c_1(L')\right) \wedge \mathrm{Ch}_{\mathcal{A}}(E) \wedge \exp\left(\frac{1}{N}c_1(L)\right)\end{aligned}$$

where $\mathrm{Ch}_{\mathcal{A}} : K^0(Z;\mathcal{A}) \longrightarrow H^{\mathrm{even}}(Z;\mathbb{Q})$ *is the Chern character in twisted K-theory,* $c_1(L)$ *is the first Chern class of* L*, the* N*th power of the line bundle* $\tilde{L}$ *over* $\mathcal{P}$ *and* $c_1(L')$ *is the first Chern class of* L'*, the square of the line bundle* $\tilde{L}'$ *over* $\mathcal{P}'$*.*

8. Index formula

Theorem 4. *Given an Azumaya bundle,* $\mathcal{A}$*, over an even dimensional compact manifold* Z*, the analytic index defines a map*

$$\mathrm{ind}_a : K^0_c(T^*Z;\pi^*\mathcal{A}) \longrightarrow \mathbb{Q} \tag{58}$$

where $\mathrm{ind}_a(A) = \mathrm{ind}_a(\sigma(A))$ *for elliptic elements of* $\Psi_\epsilon(Z;E,F)$ *for projective vector bundles associated to* $\mathcal{A}$ *and*

$$\mathrm{ind}_a(b) = \int_{T^*Z} \mathrm{Td}(T^*Z) \wedge \mathrm{Ch}_{\mathcal{A}}(b), \quad \forall\ b \in K_c(T^*Z;\pi^*(\mathcal{A})). \tag{59}$$

Proof. It has been shown above that $\mathrm{ind_a}(A)$, for elliptic elements of $\Psi_\epsilon(Z;E,F)$ is additive, homotopy invariant and multiplicative on composition. Thus it does descend to a map as in (58), just as in the standard case, but with possibly real values. As such a real-valued additive map on the twisted K-space $K_{\mathrm{c}}(T^*Z;\pi^*(\mathcal{A}))$, $\mathrm{ind_a}$ must factor through the Chern character, since it is an isomorphism over $\mathbb{R}$ (or $\mathbb{Q}$). Thus

$$\mathrm{ind_a}(b) = \widetilde{\mathrm{ind}}_{\mathrm{a}}(\mathrm{Ch}_{\mathcal{A}})(b)),\ \widetilde{\mathrm{ind}}_{\mathrm{a}} : H^{\mathrm{even}}_{\mathrm{c}}(T^*Z;\mathbb{Q}) \longrightarrow \mathbb{R} \tag{60}$$

being a well-defined map. However we may construct such elliptic projective pseudodifferential operators by twisting the signature operator by a projective vector bundle associated to $\mathcal{A}$. For these (54) gives the index. From the Thom isomorphism in cohomology, we know that these elements generate $H^{\mathrm{even}}_{\mathrm{c}}(T^*Z;\mathbb{Q})$ so suffice to compute the map $\widetilde{\mathrm{ind}}_{\mathrm{a}}$ in (60). Thus it suffices to show that the Riemann-Roch formula (54) is consistent with (59), but this follows from the standard case of the index formula and linearity. q.e.d.

In the non-oriented case we can pass to the oriented cover and deduce the same formula. Similarly if we consider pseudodifferential operators acting between $\tilde{L}$ twisted projective vector bundles corresponding to a line bundle $\tilde{L}$ over the bundle of trivializations of an Azumaya bundle $\mathcal{A}$, and with Nth power L over the base, we arrive at the analogous twisted formula generalizing (59) and (57)

$$\mathrm{ind_a}(Q) = \int_{T^*Z} \mathrm{Td}(T^*Z) \wedge \mathrm{Ch}_{\mathcal{A}}(\sigma(Q)) \wedge \exp\left(\frac{1}{N}c_1(L)\right), \quad \forall\ Q \in \Psi_\epsilon(Z;E,F) \text{ elliptic.} \tag{61}$$

In the odd-dimensional case we may use suspension to reduce to the even-dimensional case and again arrive at (61). Namely take the exterior tensor product with an untwisted operator of index one on the circle. To do this it is necessary to generalize the discussion in Section 1 to 6 to such 'product type' operators, including the homotopy invariance, enough to show that this exterior tensor product can be deformed, through elliptic operators in the product sense, to a true (projective) elliptic pseudodifferential operator. This is essentially a smooth analogue of arguments already present in [**1**] and we forgo the details, since geometrically the even dimensional case is the more interesting one.

9. Fractions and the index formula

On an oriented even-dimensional manifold, the vanishing of W_3 is equivalent to the existence of a $\mathrm{spin}_{\mathbb{C}}$ structure (cf. [**8**]); in particular this follows if the manifold is almost complex. In the almost complex case there is no spin structure unless the canonical bundle has a square root. Nevertheless, there is always a projective spin Dirac operator and Theorem 2 applied in this case case gives the usual formula

$$\mathrm{ind}_a(\eth^+) = \int_Z \widehat{A}(Z).$$

We recall some well known examples of oriented but non-spin manifolds where $\int_Z \widehat{A}(Z)$ is a fraction, justifying the title of the paper. The simplest is $Z = \mathbb{C}P^2$, in which case $\int_Z \widehat{A}(Z) = -\frac{1}{8}$.

Also in the almost complex case with Hermitian metric, we have the $\mathrm{spin}_{\mathbb{C}}$ Dirac operator

$$\overline{\partial} + \overline{\partial}^* : \Lambda^{0,\mathrm{even}} Z \longrightarrow \Lambda^{0,\mathrm{odd}} Z. \tag{62}$$

Its index is $\int_Z \widehat{A}(Z) e^{\frac{1}{2}c_1}$ where $c_1 = c_1(Z)$ is the Chern class of the canonical line bundle. The integral is the formula for the top term in the Todd polynomial written in terms of $\widehat{A}$ and c_1.

An amusing corollary of Theorem 3 is that we can now interpret the integral as the index of the projective Dirac operator coupled to a line bundle which is a square root of the canonical bundle. Previously this interpretation was only possible when Z was itself spin, when this square root bundle exists as an ordinary line bundle on Z.

Another important class of examples is the following. Let $V^{2n}(2d+1)$ be hypersurfaces in $\mathbb{C}\mathrm{P}^{2n+1}$. That is, in the homogeneous coordinates $[Z_0, \ldots, Z_{2n+1}]$ for $\mathbb{C}\mathrm{P}^{2n+1}$,

$$\begin{aligned} V^{2n}(2d+1) \ &= \ \Big\{[Z_0, \ldots, Z_{2n+1}] \in \mathbb{C}P^{2n+1} : P(Z_0, \ldots, Z_{2n+1}) = 0, \\ &\qquad \nabla P(Z_0, \ldots, Z_{2n+1}) \neq 0, (Z_0, \ldots, Z_{2n+1}) \neq 0\Big\} \end{aligned}$$

where $P(Z_0, \ldots, Z_{2n+1})$ is a homogeneous polynomial of degree $2d+1$. Then it is known that $V^{2n}(2d+1)$ is not a spin manifold, and that

$$\int_{V^{2n}(2d+1)} \widehat{A}(V^{2n}(2d+1)) = \frac{2^{-2n}(2d+1)}{(2n+1)!} \prod_{k=1}^{n}((2d+1)^2 - (2k)^2).$$

It is straightforward to see that for $d \geq n$, the right hand side is equal to a non-zero fraction that is not an integer.

Note that $\mathbb{C}\mathrm{P}^2$ has positive scalar curvature and the Bochner-Lichnerowicz formula holds for the projective operator $\eth^2$, yet $\operatorname{ind}_a(\eth) = \int_Z \widehat{A}(Z) = -\frac{1}{8} \neq 0$! The usual argument, by contradiction, to the vanishing of the index, and hence $\widehat{A}$ genus, is not applicable since in the twisted case there is no notion of global section of the projective spinor bundle and therefore no way to construct harmonic spinors.

As we have observed before, Z has no $\mathrm{spin}_{\mathbb{C}}$ structure if $W_3(Z) \neq 0$. Nevertheless the projective Dirac operator exists and can have a nonzero index. We thank M.J. Hopkins for examples of Z with both $W_3(Z) \neq 0$, and $\int_Z \widehat{A}(Z) \notin \mathbb{Z}$. Here is one of his examples. Let $\mathrm{S}^2 \hookrightarrow \mathbb{C}\mathrm{P}^4$ be an embedding of degree 2. In homogeneous coordinates we can take the embedding to be $(x, y) \mapsto (x^2, y^2, xy, 0, 0)$. To do surgery on the embedded S^2, we need to verify that its complex normal bundle N is trivial as a six dimensional real vector bundle $N_{\mathbb{R}}$. It is not hard to show that $c_1(N) \in H^2(\mathbb{C}\mathrm{P}^4, \mathbb{Z}) \cong \mathbb{Z}$ is equal to -4. The obstruction to $N_{\mathbb{R}}$ being isomorphic to $\mathrm{S}^2 \times \mathbb{R}^6$ is $w_2(N_{\mathbb{R}})$. But $w_2(N_{\mathbb{R}}) = c_1(N) \bmod 2 = -4 \bmod 2 = 0$.

We can now perform the surgery. A tubular neighborhood of the embedded S^2 is $\mathrm{S}^2 \times \mathrm{Disc}^6$ with boundary $\mathrm{S}^2 \times \mathrm{S}^5$. Replace the tube by $\mathrm{Disc}^3 \times \mathrm{S}^5$ gluing its boundary $\mathrm{S}^2 \times \mathrm{S}^5$ to the tube boundary. We obtain a manifold Z that is oriented cobordant to $\mathbb{C}\mathrm{P}^4$. Hence

$$\int_Z \widehat{A}(Z) = \int_{\mathbb{C}\mathrm{P}^4} \widehat{A}(\mathbb{C}\mathrm{P}^4) = \frac{3}{128}.$$

and Z is not a spin manifold, i.e. $w_2(Z) \neq 0$. The surgery makes $H^2(Z, \mathbb{Z}) = 0$. Hence $W_3(Z) \neq 0$ from the usual long exact sequence,

$$\ldots \to H^2(Z, \mathbb{Z}) \to H^2(Z, \mathbb{Z}) \to H^2(Z, \mathbb{Z}_2) \to H^3(Z, \mathbb{Z}) \to \ldots,$$

where the first arrow is multiplication by 2.

References

[1] M.F. Atiyah & I.M. Singer, *The index of elliptic operators*, III, Ann. of Math. **87** (1968) 546–604, MR 0236952, Zbl 0164.24301.

[2] N. Berline, E. Getzler, & M. Vergne, *Heat kernels and Dirac operators*, Springer-Verlag, Berlin, 1992, MR 1215720, Zbl 0744.58001.

[3] J.-L. Brylinski, *Loop spaces, characteristic classes and geometric quantization*, Progress in Mathematics, **107**, Birkhäuser Boston, Inc., Boston, MA, 1993, MR 1197353, Zbl 0823.55002.

[4] A. Grothendieck, *Le group de Brauer* I, II, III, in 'Dix Exposes sur la cohomologie des schemas', North Holland, Amsterdam, 1968, MR 1608798, MR 1608805, MR 0244271, Zbl 0193.21503, Zbl 0198.25803, Zbl 0198.25901.

[5] V.W. Guillemin, *A new proof of Weyl's formula on the asymptotic distribution of eigenvalues*, Adv. Math. **55** (1985) 131–160, MR 0772612, Zbl 0559.58025.

[6] M.J. Hopkins & I.M. Singer, *Quadratic functions in geometry, topology, and M-theory*, J. Differential Geom. **70**(**3**) (2005) 329–452, MR 2192936.

[7] L. Hörmander, *The analysis of linear partial differential operators*, **3**, Springer-Verlag, Berlin, Heidelberg, New York, Tokyo, 1985, MR 1313500, Zbl 0601.35001.

[8] H.B. Lawson Jr. & M.-L. Michelsohn, *Spin geometry*, Princeton Mathematical Series, **38**, Princeton University Press, Princeton, NJ, 1989, MR 1031992, Zbl 0688.57001.

[9] B. Kostant, *Quantization and unitary representations,* I, *prequantization*, Lectures in modern analysis and applications, III, Lecture Notes in Math., **170**, Springer, Berlin, 1970, 87–208, MR 0294568, MR 0294568, Zbl 0223.53028.

[10] V. Mathai, R.B. Melrose, & I.M. Singer, *The index of projective families of elliptic operators*, Geom. Topol. **9** (2005) 341–373, MR 2140985.

[11] R.B. Melrose, *Star products and local line bundles*, **54**(**5**) (2004) 1581–1600, MR 2127859, Zbl 1061.47064.

[12] ———, *The Atiyah-Patodi-Singer index theorem*, A K Peters Ltd., Wellesley, MA, 1993, MR 1348401, Zbl 0796.58050.

[13] R.B. Melrose & V. Nistor, *Homology of pseudodifferential operators on manifolds with corners* I. *Manifolds with boundary*, [funct-an/9606005].

[14] R.B. Melrose & F. Rochon, *Families index for pseudodifferential operators on manifolds with boundary*, Int. Math. Res. Not. 2004, **22**, 1115–1141, MR 2041651.

[15] M.K. Murray & M.A. Singer, *Gerbes, clifford modules and the index theorem*, Ann. Global Anal. Geom. **26**(**4**) (2004) 355–367, MR 2103405.

[16] M. Wodzicki, *Noncommutative residue.* I. *Fundamentals*, in 'K-theory, arithmetic and geometry' (Moscow, 1984–1986), Lecture Notes in Math., Springer, Berlin-New York, 1987, 320–399, MR 0923140, Zbl 0649.58033.

Department of Mathematics
University of Adelaide
Adelaide 5005, Australia
E-mail address: mathai.varghese@adelaide.edu.au

Department of Mathematics
Massachusetts Institute of Technology
Cambridge, MA 02139
E-mail address: rbm@math.mit.edu

Department of Mathematics
Massachusetts Institute of Technology
Cambridge, MA 02139
E-mail address: ims@math.mit.edu